REVIEWS

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A New Kind of Science. By Stephen Wolfram. Wolfram Media, Champaign, IL, 2002, xiv + 1197 pp., ISBN 1-57955-008-8, $44.95.

Reviewed by **Rudy Rucker**[1]

Stephen Wolfram's *A New Kind of Science* offers a cornucopia of ideas that are likely to be discussed for years to come. The book has been something of a mainstream best-seller and could well bring new students and support to the fields that lie at the interface between mathematics and computer science. For a teaching professional, the book opens opportunities to develop enjoyable new courses both at high and at low levels. In addition, it describes a very large number of computer experiments cunningly designed to illustrate Wolfram's points. In every case the experiments are summarized in graphics of remarkably transparent and informative design. As well as being interesting and popular, *A New Kind of Science* is a landmark achievement in the visual presentation of scientific information.

So what's not to like?

Many reviews (such as [**7**]) have pointed out that some of the material in *A New Kind of Science* is old work done by Wolfram and others in the 1980s [**14**]. But the book contains quite a few new ideas as well. Some things that seemed new to this reviewer include the following:

- space-time diagrams of one-dimensional Turing machines (p. 78)
- complexity in the powers of three (p. 119)
- a rethinking of the role of natural selection in biological evolution (p. 383)
- an explanation for the occurrence of the golden ratio in phyllotaxis based on computer experiments with reaction-diffusion simulations (p. 410)
- a simple idealized model of market behavior (p. 429)
- steps towards a new fundamental theory of physics (chap. 9)
- a cellular automaton that generates a series of stripes with gaps matching the locations of the primes (p. 640)
- the computational universality of the cellular automaton Rule 110 (p. 675) and, as a corollary, a new lower bound on the simplest possible computation-universal Turing machine (p. 707)
- Wolfram's Principle of Computational Equivalence (p. 716)
- an explanation of free will in a deterministic world (p. 750)
- a single-axiom formulation of prepositional logic (p. 808)
- a reconsideration of the origin of mathematics (p. 791)

[1]The reviewer is grateful to the Flemish Academy of Arts and Sciences (VLAC) for providing him with support while working on this review. He would also like to acknowledge useful discussions with Leon Horsten and Brian Silverman.

For scientists, two big problems with *A New Kind of Science* are that Wolfram consistently adopts an immodest tone and that he is ungenerous in his acknowledgments of others' work. Regarding the first problem, a common feeling seems to be that Wolfram is promising us the moon while delivering little more than an explanation for the markings on a certain kind of sea shell. (See [**2**], [**3**], [**7**], and [**13**].) Certainly Wolfram oversells himself, but the real point of his book is to bring about a certain paradigm shift rather than to offer a series of hard and fast empirical predictions. Of course, this is a risky path to chart; one thinks for instance of the hype surrounding catastrophe theory and the early claims for chaos theory.

Examples of the second problem include Wolfram's legal saber-rattling to ensure that *A New Kind of Science* would contain the first publication of a certain short single axiom for prepositional logic that also appears in [**5**] and the first publication of Matthew Cook's proof of the universality of Rule 110 (see [**2**]). Wolfram also fails to explicitly ascribe some results to the persons usually associated with them, such as Gerard Vichniac's voting rule [**12**] and Norman Packard's snowflake rule [**10**]. Moreover, he often refuses to use standard nomenclature not invented by him. An example is John H. Conway's term "glider" for a persistent pattern of cells that moves about in the simulation space, which is used by every cellular automatist except Wolfram.

Of course, controversy and dispute are part of a living science. The real news here is the intellectual content of the book, not Wolfram's personality. The quality of the book is so high that one does best to enjoy and debate the ideas on their own merits, while keeping in mind that any claims of priority appearing in *A New Kind of Science* should be carefully researched.

Overview. *A New Kind of Science* presents a sustained argument for viewing our world as being made up of computations. The weaving of ripples in a flowing stream of water, the flow of our thoughts, the twisting of a tree branch seeking the light, the shower of elementary particles from a moderate-energy collision: for Wolfram, all of these are best viewed as computations of one kind or another. Rather than invoking the overworked name of Church for this viewpoint, the reviewer proposes to coin the name "Automatism" for Wolfram's thesis that "it is possible to view every process that occurs in nature or elsewhere as a computation" (p. 716).

In the past, our scientific investigations have tended to focus on finding equations to describe the world. The paradigm shift that Wolfram proposes is to focus instead on finding simple computations to model reality. This is what he means by "a new kind of science." In addition, Wolfram argues that the naturally occurring computations that we see around us fall into one of two categories: simple processes that can easily be seen to repeat or to nest themselves in an obvious way and complex processes, nearly all of which Wolfram believes to be "computation-universal" in the sense of being like universal Turing machines, which can simulate any discrete computation when given the proper input. The universality of complex processes (what Wolfram calls the "Principle of Computational Equivalence") and the related notion of computational irreducibility are important themes in Wolfram's work; we shall discuss them in detail later.

During the course of the book, Wolfram draws a number of interesting consequences from his world view. Two striking examples mentioned in the book's first pages are the following.

(1) The reason we can't understand very much of the world in a compact form is that most of the phenomena we observe are computation-universal and are therefore "computationally irreducible." (One might object here that we

can't actually prove that various natural phenomena are computation-universal. Wolfram would answer that the computation-universality of complex natural phenomena is on the order of a natural law and doesn't require case-by-case validation.)

(2) Formally undecidable statements, such as first appeared in Gödel's Incompleteness Theorem, aren't a rare kind of thing in the physical world; indeed, very many physical questions will be found to be formally undecidable from our theories of physics.

A key point about Wolfram's work is that it's experimental, although his experiments are carried out on a computer and hence (unlike physical experiments) are mathematical in nature and exactly reproducible. Wolfram's strategy is to look at arbitrary computations that use simple rule classes and start with simple initial conditions, rather than special ones constructed for particular purposes. The latter computations often involve extremely complicated rules, explicitly designed to produce the features that we want. But with increased experience, it's common to discover that a simpler rule would have worked, and a point that Wolfram repeatedly emphasizes is that we can simulate virtually any kind of targeted complex behavior by finding an appropriate simple rule.

Wolfram also points out that our standard technologies use only a very limited set of rules. When using science to design normal kinds of technology, we figure out what we want a system to do, then engineer it using the simple and predictable systems that we already know. But nature is under no such constraint. Although a remarkably small set of basic ingredients and basic structures is used in making, say, biological organisms, these structures are coupled into very large parallel-computing systems. It is common for natural systems to achieve their effects by the unpredictable accumulation and interaction of a myriad of small computational steps. There is a whole sea of heretofore unutilized computational rules that might have useful technological applications.

Simple computations. A brief definition of cellular automata is in order for nonspecialists. A cellular automaton (CA for short) is a process that operates on a cell-space that is, for Wolfram, typically one-dimensional. The cell space is divided into discrete cells, and each cell contains a symbol called its value. A CA has a parallel update process by which the value of every cell in the cell space is simultaneously replaced by a new value. Ordinarily the CA is also required to be local and homogeneous. Locality means that a given cell's new value depends only on the present values of the cells in some fixed neighborhood of it. Homogeneity means that (i) the neighborhoods in the update process are all of the same shape and (ii) each cell uses the same recursively specified update rule.

In the very simplest CAs, the cell-space is a one-dimensional row of cells, the symbol set is 0 and 1, and each cell "sees" only itself and its nearest neighbors on the left and on the right, making a three-cell neighborhood. A CA of this simple kind can be specified by describing how a cell's new value depends on which of the eight possible three-cell neighborhood configurations it lies in. This makes for 2^8 possible rules, which are conventionally labeled by integers from 0 to 255.

Wolfram's first investigations in the 1980s involved starting each of the basic 256 rules on a row full of 0s with a single cell set to 1 and observing what classes of behavior occur. For example, Rule 254 "dies out" or becomes uniform, Rule 250 generates a striped or periodic pattern, Rule 90 generates a nested Sierṕinski triangle pattern, Rule 30 is random-looking, and Rule 110 has persistent local structures (gliders). It turns out that these same few classes of behavior arise if we look at more complicated

CAs, such as those that allow more than two symbols, those that look at more than the very nearest neighbors, or those that use higher dimensional cell-spaces.

Wolfram identifies the observed CA behaviors with four classes of behavior for possible computations:

Class 1: Dies out or becomes uniform (like Rule 254);

Class 2: Becomes periodic or generates nested structures (like rule 250 or Rule 90);

Class 3: Generates seething, seemingly random patterns (like Rule 30);

Class 4: Shows persistent local structures (gliders) that move about (like Rule 110).

In an attempt to get a more solid insight into the differences between the four classes, Wolfram also carries out experiments where a CA rule is started on two random patterns that differ on a single cell. In Class 1 the change dies out (information is forgotten), in Class 2 the change is localized and remembered, in Class 3 the change expands like a "light cone" of one additional cell per update so that the information is communicated over long ranges, and in Class 4 the changes spread sporadically (that is, sometimes the information spreads and sometimes not). In Class 4 rules the information normally seems to spread by hitching a ride upon a glider.

Wolfram also has some discussion of two-dimensional CAs, but this work is thin and weak and gives the reader little sense of other researchers' work in this field. It would have been valuable, for instance, to see Wolfram discuss the Belousov-Zhabotinsky or BZ scroll patterns one sees in simulations of excitable media, as discussed in [**4**].

A major inspiration for Wolfram's work is his abiding sense of wonder that, starting from an initial condition of a single marked cell, something as simple as Rule 30 can generate Class 3 randomness and that Rule 110 can generate a Class 4 process resembling a complex computation. He expresses his excitement as follows. "And what I found—to my great surprise—was that despite the simplicity of their rules, the behavior of programs was often far from simple. Indeed, even some of the very simplest programs that I looked at had behavior that was as complex as anything I'd ever seen.... I have come to view [this result] as one of the more important single discoveries in the whole history of theoretical science" (p. 2).

Lest one have the impression that Wolfram feels that every computation is fundamentally a cellular automaton, his chapter 3 painstakingly examines many alternate kinds of simple computational systems and finds that they all generate patterns that fall into the same four classes he first observed with his simple CAs. The computational processes he explores include traditional Turing machines, "mobile automata," substitution rules, and tag systems. Wolfram also points out that the distribution of the primes, the digits of π, and even multiplication by three are mathematically familiar computations that can generate complex patterns from simple rules.

As Brian Silverman puts it, what Wolfram has discovered here is in some ways comparable to what Church, Kleene, and Turing found. Those logicians discovered that the notion of a computable function is the same whether defined by general recursive functions, lambda calculus, or Turing machines. Wolfram has found that the detailed appearances of computations fall into the same basic classes regardless of how the computations are formulated. Each style of computation occurs in the same four flavors: uniformity, repetition or nesting, seething complexity, and complexity with moving local structures (gliders).

Randomness. The Automatist viewpoint allows Wolfram to make good progress on some of the troublesome issues in the philosophy of science such as the origin of randomness, the arrow of time, free will in a deterministic world, and the power of natural selection. There isn't room here to summarize all these discussions, but to give the flavor, we go over in some detail what Wolfram says about randomness. He distinguishes among the following three kinds of randomness, illustrating his ideas in terms of the evolution of a one-dimensional CA that would possibly allow values of individual cells to be randomized (p. 299):

(A) The environment randomizes each cell of every row at each step.

(B) The environment randomizes the cells in the top row, which then evolve deterministically.

(C) Neither the initial condition nor the evolution is random, but "intrinsic randomness" is generated by the working of the rule, as in Rule 30. This kind of randomness wasn't recognized until recently, but it's a common phenomenon.

As examples of randomness (A), Wolfram mentions Brownian motion and electronic circuit randomizers that amplify thermal noise. He notes that the amplifiers have latency, and in fact can only extract randomness at a certain maximum rate. Moreover, the more complicated you make them the slower they are, and in the very complex ones there's a good chance that randomness (C) is having an effect as well.

Examples of randomness (B) might be flipping a coin, rolling a die, or even the jounces of a car driving down a road (the pre-existing bumps in the road being the initial conditions). In fact, one can make a reliable coin flipper, but chaos theory shows that some systems are so sensitive that we can't practically specify the initial condition precisely enough to maintain predictability for more than a few seconds. This aspect of chaos theory is sometimes dubbed "excavating endless digits from an initial condition." Wolfram has a number of issues with this notion. He points out that, in fact, any digit beyond about place ten is going to be altered by the continual interaction with the environment, so randomness (B) really reduces back to randomness (A). Wolfram's sympathies lie with randomness (C), which he calls intrinsic randomness. He argues that "whenever a large amount of randomness is produced in a short time, intrinsic randomness generation is overwhelmingly likely to be the mechanism responsible" (p. 321). The prize example of intrinsic randomness is Rule 30, which can be used to generate bit strings that are statistically random in the sense of having maximal entropy and equiprobable occurrences of same-length substrings.

Wolfram chooses to downplay the notion of so-called algorithmic or Chaitin-Kolmogorov randomness, by the way. He feels that randomness should be taken as more of an experiential or perceptual property, and that we should say a pattern fails to be random only if it has regularities that "can realistically be recognized by applying various kinds of analysis to the sequence" (p. 317). He remarks that he has used Rule 30 for many years as the randomizer for his commercial product Mathematica, running Rule 30 on an array a few hundred cells wide and wrapping it around at the edges. In the 1980s Wolfram even patented a circuit design that uses Rule 30 as a randomizer [**15**]. In chapter 10 we find a thorough but not quite conclusive analysis of the question of whether Rule 30 can be used to generate bits that satisfy the strong condition of being cryptographically random.

As a final speculation, Wolfram wonders if some "random" experiments are actually repeatable because they depend on intrinsic randomness generation rather than randomness (A) or (B). That is, the macroscopically visible patterns generated by randomness (C) may not be very sensitive to the precise initial conditions of the experiments.

Modeling the world. One of the criticisms leveled against *A New Kind of Science* (in [**3**] and [**13**], for instance) is that Wolfram does not present a detailed, predictive model for any specific real-world process. Certainly the book would be more compelling if it contained a success story on a par with accounting for the periods of the planetary orbits or the perihelion shift of Mercury. But Wolfram presents a good deal of intriguing evidence that it would liberate science to get more in the habit of using lightweight models such as cellular automata or other simple computations, rather than the traditional analytic models such as differential equations.

This is not to say, by the way, that differential equations would lose their place in the sun. After all, the finite difference methods used to simulate differential equations are really just continuous-valued cellular automata [**9**]. A familiar example of the interchangeability of differential equations and continuous-valued cellular automata arises in descriptions of simple reaction-diffusion processes that can generate markings such as one sees on animal coats, sea shells, or butterfly wings [**6**], [**11**]. Wolfram outlines some interesting new directions in this area (p. 165).

Wolfram's simulations indicate that the large-scale patterns in fluid flow (such as a Karman vortex street) are actually not sensitively dependent on initial conditions. By focusing on a CA simulation of fluid flow, he finds that the randomness in a turbulent fluid is intrinsically generated, that is, of the randomness type (C) mentioned earlier, rather than the type (A) or (B) that one might otherwise suspect. Wolfram suggests interesting experiments to show that "with sufficiently careful preparation it should be possible to produce patterns of flow that seem quite random but that are nevertheless effectively repeatable" (p. 382). He also predicts that we should be able to find very simple programs "that will successfully manage to reproduce the main features of even the most intricate and apparently random forms of fluid flow" (p. 382).

Wolfram's comments on biology are particularly intriguing. He takes the position that "many of the most obvious examples of complexity in biological systems actually have very little to do with adaptation or natural selection" (p. 383). Instead the complex forms we see in living things are simply the result of the fact that "in almost any kind of system many choices of underlying rules inevitably lead to behavior of great complexity" (p. 383). Complexity isn't something rare that needs to be forced into existence via evolution; it is ubiquitous and will arise very readily in biological systems consisting, as they do, of many interacting parts.

Wolfram argues that, rather than resulting from optimization, the complex forms and behaviors of living organisms happen to be reasonably easy patterns to generate that serve the organism well enough to get by. As a simple example, the exact patterns of color on a butterfly's wings needn't be encoded in the creature's genome. The genetic code merely sets up a CA-like reaction-diffusion process (or finite-difference-method-simulated differential equation, if you will) that generates patterns that serve reasonably well for, say, camouflage or sexual attractiveness. "In the past, the idea of optimization for some sophisticated purpose seemed to be the only conceivable explanation for the level of complexity that is seen in many biological systems. But with the discovery ... that it takes only a simple program to produce behavior of great complexity, a quite different—and ultimately much more predictive—kind of explanation immediately becomes possible" (p. 388). In point of fact, Wolfram's methods really can't predict all that much yet. But they point the way to a sunny future in which we can understand gene codes as programs for simple computational processes.

In chapter 9, Wolfram provides a refreshingly clear discussion of the Second Law of Thermodynamics. How is it that microphysics is time-reversible, yet macroscopic systems show increasing entropy? And how is that biological systems manage to fight off the Second Law and maintain a high level of organization? He uses time-reversible 1D cellular automata to illustrate the standard answer to the first question: that one has essentially a zero probability of setting up an initial condition that would produce an effect such as shards jumping off a floor to produce an intact glass. Regarding the second question, Wolfram has discovered a particular time-reversible CA that seems not to obey the Second Law, that is, it generates systems in which "membranes" close off subsystems that maintain a substantial degree of order.

The bulk of chapter 9 is devoted to what is surely the Holy Grail of Automatists such as Wolfram and Edward Fredkin:[2] discovering a simple computational process capable of generating the actual universe in which we live. Wolfram tackles some of the more obvious problems with this world view, but, perhaps unsurprisingly, leaves the quest unresolved. One point worth mentioning is that Wolfram is certainly not trying to say that the universe is a cellular automaton. Instead he draws upon two other kinds of simple computation: evolving networks and substitution systems. He shows how to use these seemingly unpromising beginnings to develop something like space and time without committing the solecism of embedding his computations in space and time to begin with.

Universal computation. Chapter 11 is mainly about the proof by Matthew Cook and Wolfram that CA Rule 110 is computation-universal. The proof sketched depends on showing that Rule 110 can simulate a so-called cyclic tag rule that in turn simulates a tag rule that simulates a Turing machine. At the end of the proof sketch, the reviewer would like to have seen a clear description of how to set up Rule 110 directly to act as a simulation of an arbitrary Turing machine.

The closing chapter 12 presents Wolfram's "Principle of Computational Equivalence" (PCE for short). Wolfram introduces the notion with this fanfare. "Among Principles in science the Principle of Computational Equivalence is almost unprecedentedly broad—for it applies to essentially any process of any kind, either natural or artificial. And its implications are both broad and deep, addressing a host of longstanding issues not only in science, but also in mathematics, philosophy and elsewhere" (p. 715).

Here is Wolfram's preferred statement of his PCE, taken from page 716:

> Almost all processes that are not obviously simple can be viewed as computations of equivalent sophistication.

If we also adopt the Automatist belief that everything is a computation, the PCE has very wide applicability. Let's try to make a precise mathematical statement of the PCE so as to understand better what it might mean.

What kinds of computations are "obviously simple"? At times Wolfram sounds as if he wants "obviously simple" to be restricted to mean "periodic or nested." It might make sense, however, to use the notion of "a Turing machine with a solvable halting problem" as a precise condition for "simplicity." (Note that using a broad notion of "obviously simple" only makes the PCE likelier to be true.)

How about "of equivalent sophistication"? Here the most reasonable notion to use is "having the same Turing degree," as used in recursion theory. And given that some computations are universal, and that any computation with the same Turing degree as a universal computation is universal as well, it turns out that "of equivalent sophistication" might just as well mean "universal."

[2]See http://www.digitalphilosophy.org.

So now it would seem that what the PCE says is the following:

> Every Turing machine with an unsolvable halting problem is universal.

There's an immediate problem: recursion theorists have known since the 1950s that this is false! (See the discussion of Post's problem in [**8**, pp. 304ff.].) Actually the situation is even worse: not only are the Turing degrees densely ordered, they fail to be linearly ordered.

Wolfram is of course aware of this snag, and he has two kinds of responses to it. He has carefully phrased his PCE so that it has two loopholes, which are to be found in the first and last words of its statement. Filling these loopholes, we get something more like this:

> Almost every commonly occurring unsolvable Turing machine is either universal or has a construction based on a universal Turing machine.

Regarding the first loophole, Wolfram is saying that unsolvable nonuniversal Turing machines "almost never" occur in natural contexts. This is probably the most interesting aspect of the PCE, in that it seems to say something about the kinds of processes that actually occur in the real world. And the second loophole says that the processes by which recursion theorists construct their unsolvable nonuniversal Turing machines always depend so heavily on the use of universal Turing machines that the constructed machines are "of equivalent sophistication" to universal Turing machines. But Wolfram's discussion of this claim on page 1130 isn't very convincing.

A historical analogy may be relevant. The first constructions of transcendental real numbers were carried out by Joseph Liouville in 1844. However, Liouville's numbers, such as $\sum_1^\infty 10^{-n!}$, are quite artificial. Someone might argue that a number like this is unlikely to occur in any real context. In 1874, Georg Cantor constructed a specific enumeration of the countable set of algebraic reals and produced a transcendental number by requiring it to differ in the ith digit from the ith algebraic number of his enumeration. Again, someone could say that Cantor's number is artificial and depends in an essential way upon higher-order concepts such as treating an infinite enumeration of reals as a completed object. But of course in 1873 Charles Hermite proved that the commonly occurring number e is transcendental, and in 1882 Ferdinand Lindemann proved that even π is transcendental.

So it seems at least plausible that there may in fact be naturally occurring processes of intermediate Turing degree. Of particular interest would be the question, which Wolfram implicitly raises, of whether the "random" Rule 30 might be of intermediate Turing degree. Wolfram himself thinks that the rule can eventually be proved to be universal, but it would certainly be interesting if someone could find a proof that it is not. A related concept important to Wolfram is "computational irreducibility." This term, which Wolfram does not formally define, describes computation-like processes for which there is no fast way to predict the result and whose outcome can be found only by actually simulating every step of their evolution. For example, Wolfram believes that a computation such as Rule 30 is irreducible in the sense that there is no short formula to predict, say, whether the center pixel of the nth update row is 0 or 1. The relevance of this notion hinges upon the idea that universal computations are computationally irreducible, and hence that (by the PCE) for most physical processes there will not be any quick and clean way to predict an outcome.

Wolfram sets this situation in opposition to the successes of traditional science, such as using Newton's laws plus a modest amount of computation to predict the position of a particle quite some time into the future. When we use Newton's laws, we predict the future states of a dynamical system by drawing on closed-form solutions to the differential equations of motion. Instead of running a finite-difference-method through some power of n steps, we simply do a few elementary arithmetic operations involving the digits of n, which is a computation of order $\log n$.

Wolfram feels that the successes of science are limited to a small number of phenomena, and that in most situations we will have to fall back on the computationally intensive process of simulating the evolution of the systems about which we want to make predictions. Indeed, most natural processes can't be predicted in detail by any simple formula—if for no other reason than that there are so many processes and so few simple formulas that there aren't enough "elegant laws of nature" to go around! (Gregory Chaitin has used a very similar counting argument to show that most integers n must be Chaitin-Kolmogorov random in the sense of not being equal to $T_e(0)$ for any Turing program code with $e < n$; see [**1**].)

Plausible though the notion may seem, there are serious difficulties in formalizing Wolfram's notion of "computational irreducibility." If one sets no limit on the complexity of the computing machines one is talking about, then no computation is "computationally irreducible," for any computation can be done faster using a machine with more states in order to simulate the target computation "in chunks." Stephen Weinberg tellingly makes this point: "This is why Dell and Compaq don't sell Turing machines or rule 110 cellular automata" [**13**, p. 46]. To this Wolfram could perhaps reply that using a better machine divides computation time only by a certain fixed factor, and what he really means by an "irreducible computation" is one that doesn't admit the logarithmic kind of speed reduction one gets by using instead a closed-form solution to a differential equation. But he doesn't discuss the difficulties with his notion, and his treatment of the topic remains unsatisfying.

The foundations of mathematics. In the last pages of chapter 12, Wolfram turns to the foundations of mathematics. He offers an idiosyncratic explanation for Wigner's "unreasonable effectiveness of mathematics in the natural sciences." Being computational processes, mathematics and physics are both, thanks to the PCE, universal computations, "and thus show all sorts of similar phenomena" (p. 775). This is clever, but not really satisfying. Even after we're told that physics and the human mind are equivalent in their computational sophistication, we still don't quite see why, for instance, π occurs in contexts that have nothing to do with circles.

Wolfram remarks that we can regard mathematics as a "multiway system" whereby one transforms bitstrings (axioms) according to certain rules (logic). As one can choose among various applicable transformations at each step, the result is a great branching tree of possible outcomes (theorems). This is, of course, a purely formalistic view of mathematics, a straw man that most practicing mathematicians would not regard as a correct representation of their field. And now Wolfram prepares to set the straw man on fire. He proposes that we regard mathematics as one among many arbitrary multiway systems that are universal in the sense of being able to model all possible computations, and he suggests that our choice of this system rather than any other is the result of a series of historical accidents.

Naturally, no mathematician will be happy with this characterization. In practice, mathematicians don't function like formal automata that blindly and mechanically derive results; they utilize complex mental images—sometimes visual, sometimes kinesthetic, sometimes wholly abstract. In doing this, they are in some sense like a machine with richer sets of states that can effectively short-cut the work of low-level logic. Wolfram's failure to appreciate this fact adequately may be related to his problems in formulating a proper definition of computational irreducibility. In any case, his discussion of these matters is invigorating and forces one to rethink one's views.

A consequence of the universality of mathematics is that it must perforce be plagued by undecidability. Wolfram displays a table of some of the simplest possible Diophantine equations, distinguishing between those known to have integer solutions, those known to have no integer solutions, and those for which the question is still open. He thinks it likely that for some of these, such as $x^2 = y^5 + 6y + 3$, it will turn out that (i) there are no integers x and y satisfying the equation, so we will never have a proof by example that such x and y exist, and (ii) the nonexistence of such x and y will also be unprovable from existing mathematics (p. 790). A moment's thought shows that in these cases we'll never even be able to prove the question to be undecidable. How annoying!

This would be an example of how "in general undecidability and unprovability will start to occur in practically any area of mathematics almost as soon as one goes beyond the level of questions that are easy to answer" (p. 791). Certainly there will always be difficult and interesting theorems that mathematicians are able to prove, but sprinkled in among these theorems will be large numbers of simple but undecidable sentences. Set theorists are accustomed to this kind of thing, but Wolfram brings undecidability closer to home.

At this point it's useful to return to our historical analogy involving transcendental numbers, recasting it in a fashion suggested to the reviewer by Brian Silverman. Kurt Gödel's 1930 discovery of the existence of a highly artificial undecidable sentence in mathematics could be compared to Liouville's construction of his odd transcendental number. The Church-Turing proofs that our formal theories of mathematics list theorems like Turing machines with unsolvable halting problems implied the existence of an infinite number of undecidable sentences, in analogy with Cantor's proof of the existence of infinitely many transcendentals. The solution of Hilbert's Tenth Problem by Martin Davis, Yuri Mateyasivich, and Julia Robinson, especially as refined by Gregory Chaitin, led to more and more specific kinds of mathematically undecidable sentences, in the form of Diophantine equations. This might be compared to Hermite's proof that the specific number e is transcendental. And now Wolfram's PCE proposes that incompleteness and undecidability are all around us in the natural world, as immediate as π.

Should this be taken to mean that mathematicians should give up? To abandon all hope of lighting candles and to worship the darkness instead? Not at all. If anything, there's more work than ever to do. The fact that some of our familiar questions may turn out to be undecidable on the basis of our existing axioms simply makes the game more complex. And, if we take Wolfram seriously, the study of simple computations is a broad and interesting new field in its own right that has the pleasing property of involving a large amount of empirical work along with the theoretical.

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Ramanujan: Essays and Surveys. Edited by Bruce C. Berndt and Robert A. Rankin. American Mathematical Society (History of Mathematics Series, vol. 22), Providence, 2001, xvi + 347 pp., ISBN 0-8218-2624-7, $79.

Reviewed by **Krishnaswami Alladi**

How much can be written about the Indian mathematical genius Srinivasa Ramanujan? Quite a bit, for a variety of reasons. First, Ramanujan's mathematical accomplishments are astonishing, and with time we are getting a better understanding of their significance and depth. Next, the fact that Ramanujan made such spectacular discoveries without much formal training makes one wonder how his brilliant mind worked and what motivated his insights. Finally, Ramanujan's life story is a mixture of success and failure, (mathematical) romance and sadness, the mystery of the East and the sophistication of the West. All these things make him a fascinating character to study, with the result that there is a multitude of articles describing various aspects of his life as well as research papers that build on his pathbreaking discoveries. The book under review assembles a fine collection of essays by several distinguished authors on a variety of topics that range over Ramanujan's remarkable life and contributions, his constant struggle with various illnesses in India and in England (none of which dulled his ferocious productivity), his manuscripts and notebooks, and individuals who played an important role in his life and rendered timely help. This book is a worthy sequel to *Ramanujan: Letters and Commentary* [**2**] (see my review [**1**]). It also nicely complements Hardy's classic *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work* [**3**] and Ramanujan's *Collected Papers* [**5**].

The book opens with a set of four photographs of Ramanujan and an analysis of each photograph. The best known of these is the passport photo of Ramanujan taken when he sailed to England in 1914 to work with G. H. Hardy in Cambridge University. After Ramanujan's death in 1920, Hardy wanted a photograph of Ramanujan in connection with the publication of his collected papers. As related in the book, the young astrophysicist S. Chandrasekhar (who won the Nobel Prize in 1983) was traveling to India, and Hardy requested him to secure a photograph of Ramanujan while he was there. Chandrasekhar met Ramanujan's widow Janaki and was delighted to find that